Euler coordinates for X(n)-X(m) on the Brocard axis

The Brocard axis has the linear equation given by

$$(S_A^2+S^2)(S_B-S_C)x+(S_B^2+S^2)(S_C-S_A)y+(S_C^2+S^2)(S_A-S_B)z=0$$

(see **Tables** "Euler coordinates for infinity points not on the Euler line"), and the triangle centers on the Brocard axis are listed in **Tables** "Central lines".

The triangle centers X(x : y : z) on the Brocard axis can generally be expressed as

$$x=K_0(S^2-S_BS_C)+K_1\{(E+F)-S_A\}+K_2\{(E+F)^2-S_A^2\}$$
$$y=K_0(S^2-S_CS_A)+K_1\{(E+F)-S_B\}+K_2\{(E+F)^2-S_B^2\}$$
$$z=K_0(S^2-S_AS_B)+K_1\{(E+F)-S_C\}+K_2\{(E+F)^2-S_C^2\}$$

up to the second order in (S_A, S_B, S_C), where the coefficients K_0, K_1, K_2 are symmetric functions of a, b, c. K_0, K_1 and K_2 for the triangle centers X(n) (n $<$1000) are shown in Table 1.

Hereafter, K_0, K_1, K_2 for X(n) are written as X(n) = (K_0, K_1, K_2) similar to the Shinagawa coefficients (G, H) on the Euler line.

Since X(3)= (1, 0, 0), X(6) = (0, 1, 0) and X(39) = (0, 0, 1), X(n) is expressed as

$$X(n)=K_0\,X(3)+K_1\,X(6)+K_2\,X(39).$$

Hence, the Euler coordinates ($x_e(n)$, $y_e(n)$) for X(n) = (K_0, K_1, K_2) are given by those for X(3), X(6) and X(39) as shown below,

$$x_e(n)=[K_0'xe(3)+K_1'xe(6)+K_2'xe(39)]/N,$$
$$y_e(n)=[K_0'ye(3)+K_1'ye(6)+K_2'ye(39)]/N,$$
$$N=K_0'+K_1'+K_2'$$

where

$$K_0'=\{2S^2\}K_0,\quad K_1'=\{2(E+F)S\}K_1,\quad K_2'=\{2(E+F)^2+S^2\}K_2$$

Here, the Euler coordinates for X(3), X(6) and X(39) are shown in Table 2, For example, the Euler coordinates for X(15) = $((3)^{1/2}$, S, 0) are given as

$$xe(15)=[(3)^{1/2}(2S^2)(E-2F)/2+\{2(E+F)S\}S^2/(E+F)]/[2(3)^{1/2}S^2+2(E+F)S\}]$$

$= \{(3)^{1/2} (E - 2F)S + 2S^2\}/[2\{(3)^{1/2} S+(E+F)\}]$

$ye(15) = [[(3)^{1/2} (2S^2)(0)+ 2(E+F)S\{-\$S_A^2(S_B-S_C)\$/\{2(E+F)\}]/ [2(3)^{1/2} S^2+2(E+F)S\}$

$= -\$S_A^2(S_B-S_C)\$ / [2\{(3)^{1/2} S+(E+F)\}].$

The Euler coordinates for X(n) (n <1000) on the Brocard axis are shown in Table 2. Here, " Brocard infinity point" X(511) was defined as

$$X(511) = X(3)- X(6).$$

Hence, the Euler coordinates for X(511) are given as

$x_e(511) = xe(3) - xe(6) = (E - 2F)/2 - S^2/(E+F) = \{(E+F)(E-2F)-2S^2\}/\{2(E+F)\}$

$y_e(511) = ye(3) - ye(6) = \$S_A^2(S_B-S_C)\$/\{2(E+F)\}]$

The infinity points X(n) -X(m) on the Brocard axis are equivalent to X(511) and given as

. $X(n) - X(m) = C(n, m)X(511),$

where C(n, m) is a symmetric function of a, b, c (see **Tables** "Euler coordinates for infinity points not on the Euler line"). For example, the Euler coordinates for X(3) - X(32) are given by

$x_e(3) - x_e(32)= (E-2F)/2 - (E+4F)S^2/[2\{(E+F)^2-S^2\}]$

$= (E+F)[(E+F)(E-2F)-2S^2] /[2\{(E+F)^2-S^2\}$

$=[(E+F)^2/\{(E+F)^2-S^2\}] x_e(511)$

$y_e(3) - y_e(32) = (E+F)\$S_A^2(S_B-S_C)\$]/[2\{(E+F)^2-S^2\}]$

$=[(E+F)^2/\{(E+F)^2-S^2\}] y_e(511)$

Hence, $X(3) - X32) = [(E+F)^2/ 2\{(E+F)^2-S^2\}]X(511)$, and

$$C(3, 32) = (E+F)^2/ \{(E+F)^2-S^2\}].$$

Similarly, the coefficients C(n, m) for X(n)-X(m) can be obtained, and C(3, n) and C(6, n) are shown in Table 3..

Table 1 K_0, K_1, and K_2 for the triangle centers X(n) (n < 1000) on the Brocard axis : $X(n) = (K_0(S^2-S_BS_C)+K_1\{(E+F)-S_A\}+K_2\{(E+F)^2-S_A^2\} ::)$

X(*)	K_0(a,b,c)	K_1(a,b,c)	K_2(a,b,c)
X(3)	1	0	0
X(6)	0	1	0
X(15)	$(3)^{1/2}$	S	0
X(16)	$-(3)^{1/2}$	S	0
X(32)	2	0	-1
X(39)	0	0	1
X(50)	5	-3F	-1
X(52)	E+2F	$-2S^2$	0
X(58)	2	-\$ab\$	-1
X(61)	1	$(3)^{1/2}S$	0
X(62)	1	$-(3)^{1/2}S$	0
X(182)	E+F	S^2	0
X(187)	3	-E-F	0
X(216)	1	F	0
X(284)	s	(r+2R)S	0
X(371)	1	S	0
X(372)	1	-S	0
X(386)	0	\$ab\$	1
X(389)	F	$-S^2$	0
X(500)	\$aS_A\$+2abc	\$a\$$S^2$	0
X(511)	E+F	$-S^2$	0
X(566)	4	E+4F	0
X(567)	E+4F	$4S^2$	0
X(568)	E+4F	$-4S^2$	0
X(569)	E+2F	$2S^2$	0
X(570)	4	3E+4F	-2
X(571)	2	-E-2F	0
X(572)	s	rS	0
X(573)	s	-rS	0
X(574)	3	E+F	0
X(575)	E+F	$3S^2$	0

X(576)	E+F	$-3S^2$	0
X(577)	1	-F	0
X(578)	F	S^2	0
X(579)	s	-(r+2R)S	0
X(580)	$abc+\$aS_A\$$	$-\$a\$S^2$	0
X(581)	$abc+\$aS_A\$$	$\$a\$S^2$	0
X(582)	$2abc+\$aS_A\$$	$-\$a\$S^2$	0
X(583)	s	(r-3R)S	0
X(584)	s	(r+3R)S	0
X(800)	2	2F	-1
X(970)	$(E+F)\$a\$+2abc$	$-\$a\$S^2$	0
X(991)	s	(r+4R)S	0

Table 2 Euler coordinates for the triangle centers X(n) on the Brocard axis

X(*)	$x_e(n)$	$y_e(n)/\ \$S_A^2(S_B-S_C)\$$
X(3)	(E-2F)/2	0
X(6)	$S^2/(E+F)$	-1/{2(E+F)}
X(15)	$\{(3)^{1/2}(E-2F)S+2S^2\}/[2\{(E+F)+(3)^{1/2}S\}]$	$-1/[2\{(E+F)+(3)^{1/2}S\}]$
X(16)	$-\{(3)^{1/2}(E-2F)S-2S^2\}/[2\{(E+F)-(3)^{1/2}S\}]$	$-1/[2\{(E+F)-(3)^{1/2}S\}]$
X(32)	$(E+4F)S^2/[2\{(E+F)^2-S^2\}]$	$-(E+F)/[2\{(E+F)^2-S^2\}]$
X(39)	$3ES^2/[2\{(E+F)^2+S^2\}]$	$-(E+F)/[2\{(E+F)^2+S^2]$
X(50)	$-2(E-8F)S^2/[2\{(E+F)(E+4F)-4S^2\}$	$-(E+4F)/[2\{(E+F)(E+4F)-4S^2\}$
X(52)	$-\{(E^2-4F^2)-4S^2\}/(2E)$	-1/E
X(58)	$\{(E+4F)+2\$ab\$\}S^2/[2\{(E+F)^2-S^2+(E+F)\$ab\$\}]$	$-\{(E+F)+\$ab\$\}$ $/[2\{(E+F)^2-S^2+(E+F)\$ab\$\}]$
X(61)	$\{(E-2F)S+2(3)^{1/2}S^2\}/[2\{(3)^{1/2}(E+F)+S\}]$	$-(3)^{1/2}/[2\{(3)^{1/2}(E+F)+S\}]$
X(62)	$-\{(E-2F)S-2(3)^{1/2}S^2\}/[2\{(3)^{1/2}(E+F)-S\}]$	$-(3)^{1/2}/[2\{(3)^{1/2}(E+F)-S\}]$
X(182)	$\{(E+F)(E-2F)+2S^2\}/\{4(E+F)\}$	-1/{4(E+F)}
X(187)	$-(E-8F)S^2/[2\{(E+F)^2-3S^2\}]$	$-(E+F)/[2\{(E+F)^2-3S^2\}]$
X(216)	$ES^2/[2\{(E+F)F+S^2\}$	$-F/[2\{(E+F)F+S^2\}$
X(284)	{s(E-2F)+2(r+2R)S}S/[2{(r+2R)(E+F)+sS}]	-(r+2R)/[2{(r+2R)(E+F)+sS}
X(371)	{(E-2F)+2S)}S/[2{(E+F)+S}]	-1/[2{(E+F)+S}]
X(372)	-{(E-2F)-2S)}S/[2{(E+F)-S}]	-1/[2{(E+F)-S}]

X(386)	(3E+2ab)S^2/[2{(E+F)^2+S^2+(E+F)ab}]	-{(E+F)+ab} /[2{(E+F)^2+S^2+(E+F)ab}]
X(389)	-{(E-2F)F-2S^2}/(2E)	-1/(2E)
X(500)	{2aS^2+2(E-2F)abc+(E-2F)aS_A} /[2{(E+F)a+2abc+aS_A}]	-a /[2{(E+F)a+2abc+aS_A}]
X(511)	{(E+F)(E-2F)-2S^2}/{2(E+F)}	1/{2(E+F)}
X(566)	3ES^2/{(E+F)(E+4F)+4S^2}	-(E+4F)/{(E+F)(E+4F)+4S^2
X(567)	{(E+4F)(E-2F)+8S^2}/{2(5E+8F)}	-2/(5E+8F)
X(568)	-{(E+4F)(E-2F)-8S^2}/(6E)	-2/(3E)
X(569)	{(E^2-4F^2)+4S^2}/{2(3E+4F)}	-1/(3E+4F)
X(570)	5ES^2/{(E+F)(3E+4F)+4S^2}	-(3E+4F) /[2{(E+F)(3E+4F)+4S^2}]
X(571)	4FS^2/{(E+F)(E+2F)-2S^2}	-(E+2F) /[2{(E+F)(E+2F)-2S^2}]
X(572)	{s(E-2F)+2rS}S/[2r(E+F)+sS]]	-r/[2r(E+F)+sS]]
X(573)	-{s(E-2F)-2rS}S/[2{r(E+F)-sS}]	r/[2{r(E+F)-sS}]
X(574)	(5E-4F)S^2/[2{(E+F)^2+3S^2}]	-(E+F)/[2{(E+F)^2+3S^2}]
X(575)	{(E+F)(E-2F)+6S^2}/{8(E+F)}	-3/{8(E+F)}
X(576)	-{(E+F)(E-2F)-6S^2}/{4(E+F)}	-3/{4(E+F)}
X(577)	-(E-4F)S^2/[2{(E+F)F-S^2}]	-F/[2{(E+F)F-S^2}]
X(578)	{(E-2F)F+2S^2}/[2(E+2F)]	-1/[2(E+2F)]
X(579)	-{s(E-2F)-2(r+2R)S}S/[2(r+2R)(E+F)-sS)]	-(r+2R) /[2(r+2R)(E+F)-sS)]
X(580)	-{(abc+aS_A)(E-2F)-2aS^2} /[2{(E+F)a-abc-aS_A}]	-a/[2{(E+F)a-abc-aS_A}]
X(581)	{(abc+aS_A)(E-2F)+2aS^2} /[2{(E+F)a+abc+aS_A}]	-a/[2{(E+F)a+abc+aS_A}]
X(582)	-{(2abc+aS_A)(E-2F)-2aS^2} /[2{(E+F)a-2abc-aS_A}]	-a/[2{(E+F)a-2abc-aS_A}]
X(583)	{s(E-2F)+2(r-3R)S}S/[2{(r-3R)(E+F)+sS}]	-(r-3R) /[2{(r-3R)(E+F)+sS}]
X(584)	{s(E-2F)+2(r+3R)S}S/[2{(r+3R)(E+F)+sS}]	-(r+3R)/[2{(r+3R)(E+F)+sS}]
X(800)	ES^2/[2{(E^2-F^2}-S^2]]	-(E-F) /[2{(E^2-F^2}-S^2]]
X(970)	{(E+F)(E-2F)a-2aS^2+2(E-2F)abc}/(4abc)	1/(4abc)
X(991)	{s(E-2F)+2(r+4R)S}S/[2{(r+4R)(E+F)+sS}]	-(r+4R) /[2{(r+4R)(E+F)+sS}]

Table 3 C(n, m) defined by X(n)-X(m) = C(n, m)X(511) (n = 3, 6) on the Brocard axis

X(m)	C(3, m)	C(6, m)
X(3)	----	-1
X(6)	1	---
X(15)	$(E+F)/\{(E+F)+(3)^{1/2}S\}$	$-(3)^{1/2}S/\{(E+F)+(3)^{1/2}S\}$
X(16)	$(E+F)/\{(E+F)-(3)^{1/2}S\}]$	$(3)^{1/2}S/\{(E+F)+(3)^{1/2}S\}$
X(32)	$(E+F)^2/\{(E+F)^2-S^2\}$	$S^2/\{(E+F)^2-S^2\}$
X(39)	$(E+F)^2/\{(E+F)^2+S^2\}$	$-S^2/\{(E+F)^2-S^2\}$
X(50)	$(E+F)(E+4F)/\{(E+F)(E+4F)-4S^2\}$	$4S^2/\{(E+F)(E+4F)-4S^2\}$
X(52)	2(E+F)/E	(E+2F)/E
X(58)	$(E+F)\{(E+F)+\$ab\$\}$ $/[\{(E+F)^2-S^2+(E+F)\$ab\$\}]$	$S^2/[\{(E+F)^2-S^2+(E+F)\$ab\$\}]$
X(61)	$(3)^{1/2}(E+F)/\{(3)^{1/2}(E+F)+S\}$	$-S/\{(3)^{1/2}(E+F)+S\}$
X(62)	$(3)^{1/2}(E+F)/\{(3)^{1/2}(E+F)-S\}$	$S/\{(3)^{1/2}(E+F)+S\}$
X(182)	1/2	-1/2
X(187)	$(E+F)^2/\{(E+F)^2-3S^2\}$	$3S^2/\{(E+F)^2-3S^2\}$
X(216)	$(E+F)F/\{(E+F)F+S^2\}$	$-S^2/\{(E+F)F+S^2\}$
X(284)	(r+2R)(E+F)/{(r+2R)(E+F)+sS}	-sS/{(r+2R)(E+F)+sS}
X(371)	(E+F)/{(E+F)+S}	-S/{(E+F)+S}
X(372)	(E+F)/{(E+F)-S}	S/{(E+F)-S}
X(386)	$\{(E+F)^2+(E+F)\$ab\$\}$ $/\{(E+F)^2+S^2+(E+F)\$ab\$\}$	$-S^2/\{(E+F)^2+S^2+(E+F)\$ab\$\}$
X(389)	(E+F)/E	F/E
X(500)	$(E+F)\$a\$/\{(E+F)\$a\$+2abc+\$aS_A\$\}$	$-(E+F)(2abc+\$aS_A\$)$ $/\{(E+F)\$a\$+2abc+\$aS_A\$\}$
X(566)	$(E+F)(E+4F)/\{(E+F)(E+4F)+4S^2\}$	$-4S^2/\{(E+F)(E+4F)+4S^2\}$
X(567)	4(E+F)/(5E+8F)	-4(E+4F)/(5E+8F)
X(568)	(E+F)/3E	(E+4F)/3E
X(569)	2(E+F)/(3E+4F)	(E+2F)/(3E+4F)
X(570)	$2(E+F)(3E+4F)/\{(E+F)(3E+4F)+4S^2\}$	$-4S^2/\{(E+F)(3E+4F)+4S^2\}$
X(571)	$(E+F)(E+2F)/\{(E+F)(E+2F)-2S^2\}$	$2S^2/\{(E+F)(E+2F)-2S^2\}$
X(572)	r(E+F)/{r(E+F)+sS}	-sS/{r(E+F)+sS}
X(573)	r(E+F)/[r(E+F)-sS]	sS/{r(E+F)+sS}

X(574)	$(E+F)^2/\{(E+F)^2+3S^2\}$	$-3S^2/\{(E+F)^2+3S^2\}$
X(575)	3/4	-1/4
X(576)	3/2	1/2
X(577)	$(E+F)F/\{(E+F)F-S^2\}$	$S^2/\{(E+F)F-S^2\}$
X(578)	(E+F)/(E+2F)	-F/(E+2F)
X(579)	(r+2R)(E+F)/{(r+2R)(E+F)-sS}	sS/{(r+2R)(E+F)-sS}
X(580)	$(E+F)\$a\$/\{(E+F)\$a\$-abc-\$aS_A\$\}$	$(abc+\$aS_A\$)/\{(E+F)\$a\$-abc-\$aS_A\$\}$
X(581)	$(E+F)\$a\$/\{(E+F)\$a\$+abc+\$aS_A\$\}$	$(abc+\$aS_A\$)/\{(E+F)\$a\$+abc+\$aS_A\$$
X(582)	$(E+F)\$a\$/\{(E+F)\$a\$-2abc-\$aS_A\$\}$	$(2abc+\$aS_A\$)/\{(E+F)\$a\$-2abc-\$aS_A\$\}$
X(583)	(r-3R)(E+F)/{(r-3R)(E+F)+sS}	-sS/{(r-3R)(E+F)+sS}
X(584)	(r+3R)(E+F)/{(r+3R)(E+F)+sS}	-sS/{(r+3R)(E+F)+sS}
X(800)	$(E^2-F^2)/\{(E^2-F^2)-S^2\}$	$S^2/\{(E^2-F^2)-S^2\}$
X(970)	$(E+F)\$a\$/(2abc)$	$\{(E+F)\$a\$+2abc\}/(2abc)$
X(991)	(r+4R)(E+F)/{(r+4R)(E+F)+sS}	-sS/{(r+4R)(E+F)+sS}